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Moment ratios for an urn model of sand compartmentalization

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Numerically solving a master equation for a recently introduced nonequilibrium urn model of sand compartmentalization, we show that the order-parameter moment ratios of the fourth and sixth order remain constant along an exactly located line of critical points. Obtained values are in very good agreement with values predicted by Brézin and Zinn-Justin for the equilibrium Ising model above the critical dimension. At the tricritical point, these ratios acquire values that also agree with a suitably extended Brézin and Zinn-Justin approach.

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The concept of universality and scale invariance plays a fundamental role in the theory of critical phenomena [1]. It is well known that at criticality the system is characterized by critical exponents. Calculation of these exponents for the dimension of the system d lower than the so-called critical dimension d_c is a highly nontrivial task [2]. On the other hand, for $d>d_c$, the behavior of a given system is much simpler, and critical exponents take mean-field values that are usually simple fractional numbers.

However, not everything is clearly understood above the critical dimension. One example is the Ising model ($d_c=4$), where, despite intensive research, discrepancies between analytical [3] and numerical [4] calculations still persist. Of particular interest are the values of the order-parameter moment ratios at the critical point. Several years ago, Brézin and Zinn-Justin (BJ) calculated these quantities using field-theory methods [5]. Only recently are numerical simulations for the $d=5$ model able to confirm these theoretical predictions [6]. Some other properties of the Ising model above critical dimension are still poorly explained by existing theories. For example, the theoretically predicted leading corrections to the susceptibility disagree even up to the sign with numerical simulations [4].

In addition to direct simulations of the nearest-neighbor Ising model, there are also some other ways to study the critical point of the Ising model above critical dimension. For example, Luijten and Blöte used the model with $d\leqslant 3$, but with long-range interactions [7]. Using such an approach, they confirmed with good accuracy the BJ predictions for the moment ratios. As a particular case of a long-range model, Luijten and Blöte studied a model with extremely long-range interactions (the same for all pairs of spins) [8]. For this mean-field model, they were able to analytically derive the moment ratios as well as leading finite-size corrections.

In this paper, we propose yet another approach to the problem of moment ratios above critical dimension. Namely, we calculate order-parameter moment ratios of the fourth and sixth order at the critical point of a recently introduced nonequilibrium urn model [9]. Albeit structureless, this model exhibits a mean-field Ising-type symmetry breaking. Along an exactly located critical line, the obtained values are in very good agreement with values predicted by BJ. Let us notice that our calculations: (i) are not affected by the inaccuracy of the location of the critical point, which is a serious problem in the case of the Ising model; (ii) are based on the numerical solution of a discrete master equation; and (iii) are not affected by stochastic fluctuations, as in Monte Carlo simulations. Moreover, we calculate these ratios at the tricritical point, which is also present in our model, and show that the obtained values are also in agreement with suitably extended calculations of BJ. Our urn model is defined by dynamical mean-field-like rules. Although some steady-state characteristics of this model can be found exactly, their probability distributions, contrary to equilibrium systems, are unknown and can only be determined numerically. Consequently, the canonical formalism used in the equilibrium mean-field model by Luijten and Blöte [8] cannot be applied.

That both the Ising model and the nonequilibrium and structureless urn model have the same moment ratios is a manifestation of a strong universality above the upper critical dimension: at the critical point, not only the lattice structure, but the lattice itself becomes irrelevant. Moreover, a nonequilibrium nature of this model seems to be relevant as the model exhibits essentially the same type of criticality as equilibrium (mean-field) models [10]. What really matters is the type of symmetry that is broken and, since in both cases it is the same Z_2 symmetry, the equality of moment ratios follows.

Our urn model was motivated by recent experiments on the compartmentalization of shaken sand [11]. In this paper, we are not concerned with the relation with granular matter, and a more detailed justification of rules of the urn model is omitted [9]. The model is defined as follows: N particles are distributed between two urns A and B, and the number of particles in each urn is denoted as M and $N-M$, respectively. Particles in a given urn (say A) are subject to thermal fluctuations, and the temperature T of the urn depends on the number of particles in it as

$$T(x)=T_0+\Delta(1-x), \tag{1}$$

where x is a fraction of a total number of particles in a given urn and T_0 and Δ are positive constants. (For urns A and B, $x=M/N$ and $(N-M)/N$, respectively.) Next, we define dynamics of the model [9]: (i) one of the N particles is selected randomly; and (ii) with probability $\exp\{-1/[T(x)]\}$, where x

is the fraction of particles in the urn of a selected particle, the selected particle changes urns.

To measure the difference in the occupancy of the urns, we define

$$\epsilon = \frac{2M-N}{2N} = \frac{M}{N} - \frac{1}{2}. \tag{2}$$

In the steady state, the flux of particles changing their positions from A to B equals the flux from B to A. Since the selected particles are uncorrelated, the above requirement can be written as

$$\langle M \rangle \exp\left[\frac{-1}{T(\langle M/N \rangle)}\right] = \langle N-M \rangle \exp\left\{\frac{-1}{T[\langle (N-M)/N \rangle]}\right\}, \tag{3}$$

or, equivalently,

$$\left(\frac{1}{2}+\langle \epsilon \rangle\right)\exp\left[\frac{-1}{T\left(\frac{1}{2}+\langle \epsilon \rangle\right)}\right] = \left(\frac{1}{2}-\langle \epsilon \rangle\right)\exp\left[\frac{-1}{T\left(\frac{1}{2}-\langle \epsilon \rangle\right)}\right]. \tag{4}$$

Analysis of Eq. (4) shows [9] that on the (Δ, T_0) phase diagram, symmetric $(\epsilon=0)$ and asymmetric $(\epsilon \neq 0)$ solutions are separated by the critical line, which is given by the following equation:

$$T_0 = \sqrt{\Delta/2} - \Delta/2, \quad 0 < \Delta < \tfrac{2}{3}. \tag{5}$$

The critical lines terminate at the tricritical point: $\Delta = 2/3, T_0 = (\sqrt{3}-1)/3$. Let us notice that a random selection of particles basically implies the mean-field nature of this model. Consequently, at the critical point $\beta = 1/2$ and $\gamma \approx 1$ (measured from the divergence of the variance of the order parameter), which are ordinary equilibrium mean-field exponents. However, the calculation of the dynamical exponent z gives $z = 0.50\ (1)$ [9] while the mean-field value for nonconservative Ising systems is 2. We do not have convincing arguments that would explain such a small value of z. Presumably, this fact might be related to a structureless nature of our model.

Defining $p(M,t)$ as the probability that in a given urn (say A) at the time t there are M particles, the evolution of the model is described by the following master equation:

$$p(M,t+1) = \frac{N-M+1}{N} p(M-1,t)\,\omega(N-M+1) + \frac{M+1}{N} p(M+1,t)\,\omega(M+1) + p(M,t) \times \left\{\frac{M}{N}[1-\omega(M)] + \frac{N-M}{N}[1-\omega(N-M)]\right\}$$

$$\text{for } M = 1,2 \ldots N-1$$

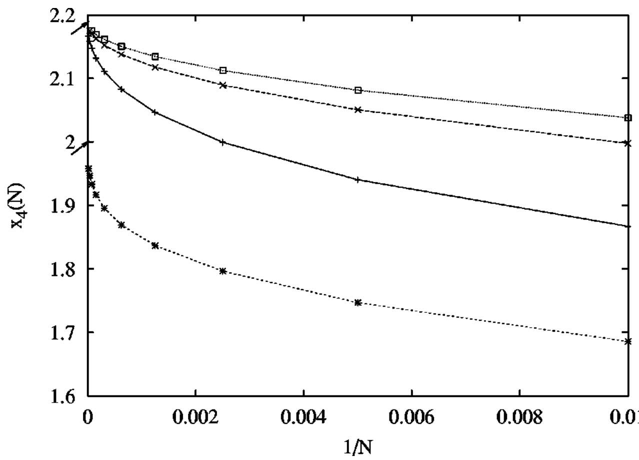

FIG. 1. The moment ratio of the fourth order $x_4(N)$ as a function of $1/N$ for (from top) $\Delta = 0.125$, 0.25, 0.5, and $\frac{2}{3}$ (tricritical point). Arrows indicate the BJ results for the critical and the tricritical point.

$$p(0,t+1) = \frac{1}{N} p(1,t)\,\omega(1) + p(0,t)[1-\omega(N)],$$

$$p(N,t+1) = \frac{1}{N} p(N-1,t)\,\omega(1) + p(N,t)[1-\omega(N)], \tag{6}$$

where

$$\omega(M) = \exp\left[\frac{-1}{T(M/N)}\right].$$

Supplementing the above equations with initial conditions, one can easily solve them numerically.

Moment ratios that we calculate are defined as

$$x_4 = \frac{\langle \epsilon^4 \rangle}{\langle \epsilon^2 \rangle^2}, \quad x_6 = \frac{\langle \epsilon^6 \rangle}{\langle \epsilon^2 \rangle^3}, \tag{7}$$

where

$$\langle \epsilon^n \rangle = \sum_{M=0}^{N} \left(\frac{M}{N} - \frac{1}{2}\right)^n p(M,\infty), \tag{8}$$

and the symbol of infinity indicates that we take the long-time (steady-state) solutions of the master equation (6). Calculations are made for $\Delta = 1/8, 1/4, 1/2$, and 2/3, and for each Δ the value of T_0 is calculated from Eq. (5). Thus, the last point is the tricritical point and the remaining ones are critical points. Numerical results for $N = 100, 200, \ldots, 51\,200$ are presented in Figs. 1–4.

Before discussing our results further, let us briefly describe the BJ approach. To calculate moment ratios above the critical dimension they used the Ginzburg-Landau-Wilson model. Then, they calculate the effective action restricting the expansion only to the homogeneous contributions (the lowest-mode approximation). Since at criticality, the quadratic (in the order parameter) term vanishes in such an ex-

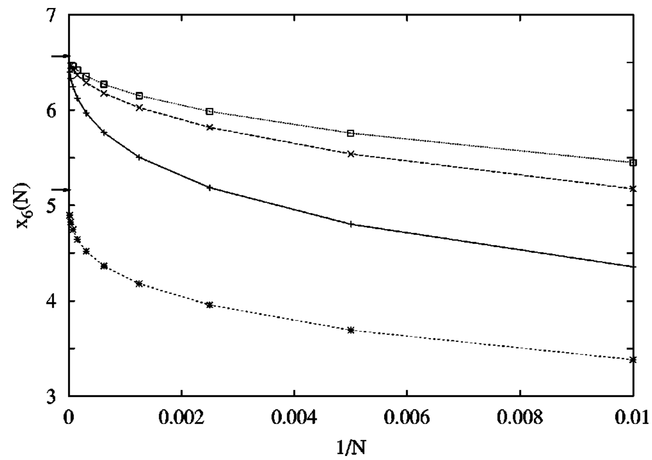

FIG. 2. The same as in Fig. 1 but for $x_6(N)$.

pansion and the leading term is quartic, which implies that the probability distribution has the form $p(x) \sim e^{-x^4}$, where x is a rescaled order parameter. Calculations of moments for such a distribution are then elementary and one obtains

$$x_4 = \frac{1}{8\pi^2}\left[\Gamma\left(\frac{1}{4}\right)\right]^4 \approx 2.188\,440\ldots,$$

$$x_6 = \frac{3}{8\pi^2}\left[\Gamma\left(\frac{1}{4}\right)\right]^4 \approx 6.565\,319\ldots. \tag{9}$$

The fact that one can restrict the expansion of the free energy to the lowest-order term is by no means obvious [3]. Such a restriction leads to the correct results only above the upper critical dimension, where the model behaves according to the mean-field scenario and fluctuations play a negligible role. For $d < d_c$, additional terms in the expansion are also important and these ratios take different values. Numerical confirmation of the above results requires extensive Monte Carlo simulations, and a satisfactory confirmation was obtained only for x_4 [6,12].

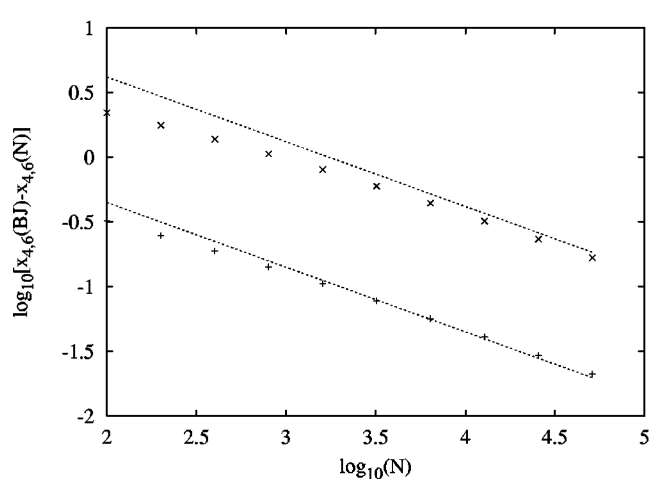

FIG. 3. Logarithmic plot of $x_4(BJ) - x_4(N)$ (+) and $x_6(BJ) - x_6(N)$ (×) as a function for N for $\Delta = 0.5$. Dotted straight lines have a slope of 0.5.

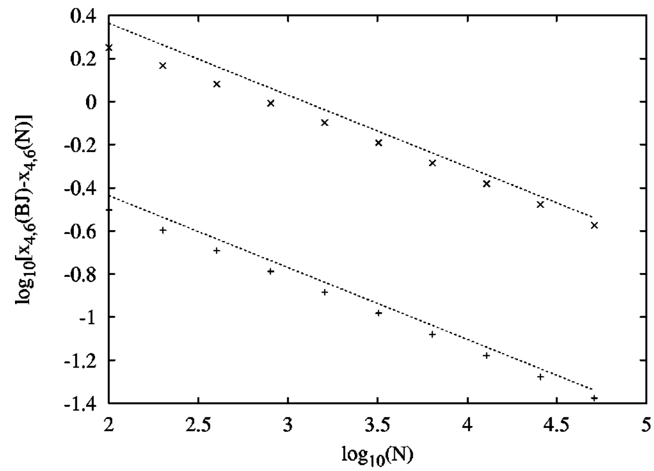

FIG. 4. Logarithmic plot of $x_4(BJ) - x_4(N)$ (+) and $x_6(BJ) - x_6(N)$ (×) as a function for N for $\Delta = \frac{2}{3}$ (tricritical point). Dotted straight lines have a slope of 1/3.

We can easily extend the BJ approach to the tricritical point at least in the lowest-mode approximation. At such a point, the quartic term also vanishes, which makes the sixth-order term the leading one and the probability distribution take the form $p(x) \sim e^{-x^6}$. Simple calculations for such a distribution yield

$$x_4 = \frac{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{2}\right)^2} = 2, \quad x_6 = \frac{\Gamma\left(\frac{1}{6}\right)^3}{6\Gamma\left(\frac{1}{2}\right)^3} \approx 5.162\,113\ldots. \tag{10}$$

The BJ results (9),(10) are indicated by small arrows in Figs. 1,2. Even without any extrapolation, one can see, especially for critical points, a good agreement with our results. Data in Figs. 1,2 shows strong finite-size corrections. To have better estimations of asymptotic values in the limit $N \to \infty$, we assume finite-size corrections of the form

$$x_{4,6}(N) = x_{4,6}(\infty) + AN^{-\omega}. \tag{11}$$

The least-squares fitting of our finite-N data to Eq. (11) gives $x_{4,6}(\infty)$, which agree with BJ values (9),(10) within the accuracy better than 0.1% (a comparable accuracy was achieved in the Monte Carlo simulations of the equilibrium mean field by Luijten and Blöte [8]). A better estimation of the correction exponent ω is obtained assuming that $x_{4,6}(\infty)$ are given by the BJ values. The exponent ω then equals the slope of the data in the logarithmic scale as presented in Figs. 3,4. Our data shows that for the critical (tricritical) point $\omega = (1/2)(1/3)$.

Let us notice that leading finite-size corrections to the moment ratios in the equilibrium mean-field model are also of the form $N^{-0.5}$ (with N being the number of spins) [8]. On the other hand, in the $d=5$ Ising model the corrections are much stronger and the leading term is of the form $L^{-0.5}$, where L is the linear system size [7]. Moreover, for the tricritical point, except for $d < d_c$, the probability distribution is known to exhibit a three-peak structure [13], which is different than the single-peak form $p(x) \sim e^{-x^6}$.

In summary, we calculated order-parameter moment ratios of the fourth and sixth order at the critical and tricritical points in an urn model that undergoes a symmetry breaking transition. Our results confirm that, as predicted by Brézin and Zinn-Justin, the critical probability distributions of the rescaled order parameter has the form $p(x) \sim e^{-x^4}$. Similarly, for the tricritical point our results suggest that $p(x) \sim e^{-x^6}$.

Although in our opinion convincing, the results are obtained using numerical methods. It would be desirable to have analytical arguments for the generation of such probability distributions. It seems that for the presented urn model this might be easier than for the (finite dimensional) Ising-type models. Let us notice that for the simplest urn model, which was introduced by Ehrenfest [14], the steady-state probability distribution can be calculated exactly in the continuum limit of the master equation, and the result has the form $p(x) \sim e^{-x^2}$, where x is now proportional to the difference of occupancy ϵ [15]. In the Ehrenfest model there is no critical point and we expect that a distribution of the type e^{-x^2} might characterize our model, but off the critical line (in the symmetric phase). We hope that when suitably extended, an analytic approach to our model might extract critical and tricritical distributions as well. Such an approach is left as a future problem.

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[1] J.L. Cardy, *Scaling and Renormalization in Statistical Physics*, (Cambridge University Press, Cambridge, UK, 1996).
[2] K.G. Wilson and J. Kogut, Phys. Rep., Phys. Lett. **12C**, 75 (1974).
[3] X.S. Chen and V. Dohm, Physica A **251**, 439 (1998); Int. J. Mod. Phys. C **9**, 1007 (1998).
[4] K. Binder, E. Luijten, M. Müller, N.B. Wilding, and H.W.J. Blöte, Physica A **281**, 112 (2000).
[5] E. Brézin and J. Zinn-Justin, Nucl. Phys. **B257**, 867 (1985).
[6] E. Luijten, K. Binder, and H.W.J. Blöte, Eur. Phys. J. B **9**, 289 (1999).
[7] E. Luijten and H.W.J. Blöte, Phys. Rev. Lett. **76**, 1557 (1996).
[8] E. Luijten and H.W.J. Blöte, Int. J. Mod. Phys. C **6**, 359 (1995).
[9] A. Lipowski and M. Droz, Phys. Rev. E **65**, 031307 (2002).
[10] U.C. Täuber, V.K. Akkineni, and J.E. Santos, Phys. Rev. Lett. **88**, 045702 (2002).
[11] H.J. Schlichting and V. Nordmeier, Math. Naturwissenschaften Unterr. **49**, 323 (1996).
[12] H.W.J. Blöte and E. Luijten, Europhys. Lett. **38**, 565 (1997).
[13] See, e.g., N.B. Wilding and P. Nielaba, Phys. Rev. E **53**, 926 (1996).
[14] P. Ehrenfest and T. Ehrenfest, *The Conceptual Foundations of the Statistical Approach in Mechanics* (Dover, New York, 1990).
[15] M. Kac and J. Logan, in *Fluctuation Phenomena*, edited by E. W. Montroll and J. L. Lebowitz (North-Holland, Amsterdam, 1987).